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Coincidence of the Shapley value and the Nucleolus in the Appointment Problem

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박 나 리

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지도교수 전 영 섭

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위 원 ________________ ㊞

Abstract

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The fixed-route traveling salesman problem with appointments, simply the appointment problem, is concerned with the following situation. A traveler makes a scheduled trip along a set of sponsors. If a sponsor in the route cancels, the traveler returns home and moves on to the next scheduled place from home. The cost of her trip has to be shared by the sponsors. We are interested in finding a way of dividing the total cost of this appointment problem among sponsors by applying solutions developed in the cooperative game theory. On this class of problems, we show the coincidence of three well-known solutions of the cooperative game theory, the nucleolus, the Shapley value, and the τ-value.

Keywords: fixed-route traveling salesman games, appointment games, Shapley value, nucleolus, τ-value, coincidence

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Contents

1 Introduction **1**

2 The model **3**
2.1 The economy 3
2.2 The appointment game 4

3 Main characterization **5**
3.1 The cost saving function 6
3.2 Coincidence of the Shapley value and the Nucleolus in the Appointment Game 9

4 Concluding remarks **15**

1 Introduction

Cost allocation problems are some important issues in economics. If a group is involved in one specific type of payment, they have to divide up the whole cost among each person. Even if everyone struggles to lessen their burden, the allocation should be done in a fair manner. Since all members of the group cover the whole cost, they need to cooperate to minimize it, and no coalition should deviate from this group and cut their cost. This is an important solution concept in cooperative game theory: the core.

Therefore, how much they dedicated to the group and the division rule matters. Here, two well-known solutions to cooperative games with transferable utilities(shortly, TU games), the Shapley value(Shapley 1953) and the Nucleolus(Schmeidler 1969), can be introduced. The Shapley value assigns to each player a payoff equal to a weighted average of her marginal contributions to all possible coalitions, with weights being determined by the sizes of coalitions. The nucleolus chooses an allocation which minimizes the difference between the worth of a coalition and its payoff in the lexicographic way.

Generally, we know that it is possible that the core is empty in TU games. Although the Shapley value can be calculated in TU games, the solution is not always in the core. However, if the core is non-empty, the nucleolus exists in the core. Then, what if the Shapley value is in the core and we can obtain the nucleolus? What difference exists between the Shapley value and the nucleolus? In this article, the fixed-route traveling salesman game with appointment (simply, the appointment game), which is introduced in Yengin(2011), is the main problem of cost allocation. The cases in the real world are introduced as follows:

We assume that each customer is to be visited exactly once but her home can be visited more than once if it is necessary, for example, when the service provider needs to replenish her supplies, or perform maintenance for the machinery/tools, after visiting a group of customers and before visiting the rest. There may also be customers who come to the main office for the service. Another reason may be that the traveler has appointments to meet with the customers and there is considerable waiting time between two consecutive appointments. Then, in between those appointments, she would go home (or to her office) and wait there.

In this appointment game, the Shapley value is calculated in simple and fascinating form[proposition 1(Yengin(2011))]. Also he gives the fact that the appointment game has a non-empty core in a given economy. Furthermore the Shapley value is in the core in this game[proposition 2(Yengin(2011)), Shapley(1971)]. Therefore, we can find the relationship between the Shapley value and the nucleolus.

The original "traveling salesman problem (TSP)" is a well-known combinatorial optimization problem in operations research. The service provider (salesman, repairman, parcel delivery guy, private tutor etc.) finds a route and decides the order of visitation simultaneously in order to travel with minimum costs. Mathematically, let $N = \{1, 2, \cdots, n\}$ denote the places to visit and *the Hamiltonian cycle* z be the complete graph with vertex set N. Then, the TSP solves the cost allocation problem in a minimum weight Hamiltonian cycle z with a condition that the vertex should be visited at least once. Note that there is no home and fixed start point (vertex) in this original TSP.

Let 0 denote *home* with agent set $N = \{1, 2, \cdots, n\}$ and let g be a directed graph with node set $N \cup \{0\}$, such that for each ordered pair $i, j \in N \cup \{0\}$ there is a directed path from i to j. Furthermore, a non-negative weight is assigned to each arc in the graph. A tour for coalition S is a round trip along the arcs in the graph of the repairman, who starts in 0, visits each customer in S at least once, and returns *home*. *Dror(1990)* shows that the core of a TSG without a home city is empty.

Potters et al.(1992) introduce the fixed route TSG which has a home. In this TSG, the order in which the players are served remains the same for all coalitions. They show that if the cost matrix satisfies the triangle inequality, and the fixed route is a minimal cost TSP route for the grand coalition, then the core is non-empty for the fixed route TSG.

Now, the fixed-route TSG with appointment has two additional representative conditions. First, if some appointments are canceled, the traveler still has to follow the initial appointment schedule due to the costs of rescheduling, inflexibility of the available or suitable times of other customers, etc. Second, if an appointment is canceled, then the traveler has to wait a lengthy period of time until the next appointment. Then, the service provider has two options: either she can wait at (or close by) the previous or next appointee's location until the next appointment starts, or she can go home (main office) and wait there. These seem like quite strong conditions, but the characteristic function is mathematically interesting. We can check the 2-person additivity which is introduced in *Chun and Hokari(2007)*, after some adjustments to the characteristic function.

This paper is organized as follows. In section 2, the general economy of a fixed route TSG is introduced and the appointment game is defined in the economy. In section 3, the main result of this article, coincidence of the Shapley value and the nucleolus in the appointment game, will be discussed. Before the main proof, the cost-saving game is introduced to deal with the benefits of agents instead of their costs. By virtue of the cost saving game, we can find the relationship between the Shapley value and the nucleolus so that setting the cost saving function is very important. Also, we get a result in which the $\tau - value$ coincides with previous solutions. Section 4 concludes with a discussion of our findings and future research directions.

2 The model

The appointment game is introduced in Yengin(2012), formally as the fixed-route traveling salesman game with appointments. We adopt the model and find the coincidence of the Shapley value and the nucleolus in this appointment game.

2.1 The economy

Let $\mathbb{N} \equiv \{1, 2, \cdots\}$ be infinitely many "potential" sponsors. Let $N \subset \mathbb{N}$ with $|N| \geq 2$ be a finite and ordered list of sponsors and the class of all such N is $\mathcal{N}$. Let 0 denote home and for each $N \in \mathcal{N}$, $N^0 \equiv N \cup \{0\}$.

The sponsors in N are listed in increasing order and the sponsors are visited in the same order as they appear in N if $i = 7, j = 9$, and $k = 3$, then $N = \{k, i, j\} = \{3, 7, 9\}$, and in an economy with sponsor set N, the first sponsor visited by the traveler is k and the last one is j.

For a given $N \in \mathcal{N}$ and $M \in \mathbb{N}$, a route $r = (i_1, i_2, \cdots, i_M)$ over N is an ordered list of the agents (sponsors and home) to be visited by a "traveler" such that

(i) the route starts from home and ends at home (i.e., $i_1 = i_M = 0$),

(ii) each sponsor is visited exactly once,

(iii) home can be visited more than once,

(iv) after sponsor $i \in N$ is visited, either home or the next highest numbered sponsor in N is visited (i.e., the relative order of the sponsors in r respect their order in N).

For each $N \in \mathcal{N}$ and each pair $\{i, j\} \subseteq N^0$, i is connected to j on a route r (denoted as $i \succ_r j$), if after i, the next agent visited is $j : r = (0, \cdots, i, j, \cdots, 0)$.

Let $c_{i,j} \geq 0$ be the cost of traveling between agents i and j. For each $i \in N$, let $c_i \equiv c_{0,i} \equiv c_{i,0}$ be the cost of traveling between home and sponsor i.

For each $N \in \mathcal{N}$, the cost of a route r over N is $c(r) = \sum_{\{i,j\} \subseteq N^0 : i \succ_r j} c_{i,j}$.

Given $N \in \mathcal{N}$, let $\mathbf{c} = \{c_{i,j} : \subseteq N^0\}$. For each N, an economy for N is $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}^N$ where $\mathcal{E} = \cup_{N \in \mathcal{N}} \mathcal{E}^N$ is the domain of all economies. For each route r over N, a sponsor set $S = \{j, k, l, \cdots, m\} \subseteq N$ is a connected set on r *if and only if* $0 \succ_r j \succ_r k \succ_r l \succ_r \cdots \succ_r m \succ_r 0$. For each $\mathbf{e} \in \mathcal{E}^N$, let $\mathcal{S}_{\mathbf{e}}$ be the set of all connected sets in economy $\mathbf{e}$. For a given $N \in \mathcal{N}$, we can associate a graph with each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle$. The elements of N^0 are called *nodes*, 0 being the *source*. A *link* between nodes i and j (denoted as l_{ij}) is a direct path between them. Let $l_i \equiv l_{0i}$ be the link between home and i. Let $L = \{l_{ij} : i, j \subseteq N^0\}$ be the set of all links between all agents in the problem. A graph g over N^0 is a subset of L. The graph associated with $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}^N$ is $g(\mathbf{e}) = \{l_{ij} : \{i, j\} \subseteq N^0 \textit{ and } i \succ_r j\}$ where each link l_{ij} in $g(\mathbf{e})$ is associated with weight $c_{i,j}$.

2.2 The appointment game

Now, the fixed-route traveling salesman game with appointments(in short, the appointment game) is formally introduced as follow.

Let $N \in \mathcal{N}$, $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}^N$, and $S \subseteq N$. Let the permissible route over S (denoted as r_S) be as follows:

For some $T \geq |S|$, let $r_S = (0, j_1, j_2, \cdots, j_T, 0)$ be such that:

(i) for each $t \in \{1, \cdots, T\}$, $j_t \in S^0$, and for each $i \in S$, there is a unique $t \in \{1, \cdots, T\}$ such that $i \equiv j_t$ on r_S,

(ii) $j_1 = \min_{i \in S} i$ and $j_T = \max_{i \in S} i$,

(iii) for each $j_t \in S$ with $t \in \{1, 2, \cdots, T\}$ and each $i \in N$ such that $j_t \succ_r i$, if $i \in S^0$, then $j_t \succ_{r_S} j_{t+1} \equiv i$,
otherwise $j_t \succ_{r_S} j_{t+1} \equiv 0$, and

(iv) for each $j_t \equiv 0$ with $t \in \{2, \cdots, T-1\}$, we have $j_t \succ_{r_S} \min\{k : k \in S \textit{ and } k > j_{t-1}\}$.

Each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}^N$, *the fixed-route traveling salesman game with appointments (in short, appointment game) associated with* **e** is

$$(N, c_{\mathbf{e}}) \text{ where } c_{\mathbf{e}} : 2^N \to \mathbb{R}_+$$

is such that for each $S \subseteq N$, $c_{\mathbf{e}}(S) = c(r_S)$. Let $\mathcal{V}_{\mathcal{E}} = \{(N, c_{\mathbf{e}}) : N \in \mathcal{N}, \mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}^N\}$ be the class of appointment games. Note that for each $N \in \mathcal{N}$ and each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}^N$, $c_{\mathbf{e}}(N) = c(r)$ and for each $S \in \mathcal{S}_{\mathbf{e}}$, $c_{\mathbf{e}}(S) = c(r_S)$. Since $c(r) = \sum_{S\in\mathcal{S}_{\mathbf{e}}} c(r_S)$, $c_{\mathbf{e}}(N) = \sum_{S\in\mathcal{S}_{\mathbf{e}}} c_{\mathbf{e}}(S)$. For each $S \subseteq N$, let $\mathbf{c}_S = \{c_{i,j} \geq 0 \ : \ \{i,j\} \subseteq S^0\}$. The economy restricted to S with respect to r_S is $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S\rangle \in \mathcal{E}^S$. Here, we denote $c_{\mathbf{e}}(\{i\}) \equiv c(i) = 2c_i$ without ambiguity.

In appointment games, a weaker condition is sufficient for the core to be non-empty. First of all, we do not need that r be a least costly tour for N. Second, we only need that given a route, for each pair of connected sponsors, the sum of their costs of connecting to home is greater than the cost of connecting to each other. Formally, for each $N \in \mathcal{N}$, each r over N, and each pair $\{i,j\} \subseteq N$ such that $i \succ_r j$, $c_i + c_j \geq c_{i,j}$. For each $N \in \mathcal{N}$, let $\mathcal{E}_T^N$ be the set of all economies with sponsor set N where this condition holds and $\mathcal{E}_T = \bigcup_{N\in\mathcal{N}} \mathcal{E}_T^N$. Let $\mathcal{V}_{\mathcal{E}_T}$ be the class of appointment games associated with economies in $\mathcal{E}_T$.

On the domain $\mathcal{V}_{\mathcal{E}_T}$, appointment games are convex and the Shapley value is in the core[Proposition 2(Yengin(2012)) by Shapley(1971)]. This shows that working on the domain $\mathcal{V}_{\mathcal{E}_T}$ is a sufficient condition for the core to be non-empty. Thus, we know that there exists a nucleolus in the game. Note that the characteristic function of the appointment game is a cost function so that the game is actually concave.

Next, given an appointment with sponsors in $S \in N$ and the permissible route over $S(r_S)$, we will prove the coincidence of the Shapley value and the nucleolus in this appointment game.

3 Main characterization

To prove the coincidence of the Shapley value and the nucleolus in appointment games, we introduce the coincidence property proposed by Chun and Hokari(2007). They showed that if the class of games satisfying two conditions[Lemma 1(Chun and Hokari(2007))], that is, every one-person-coalition's payoff is zero and the cost of a coalition with more than two people can be expressed as a sum of costs of two-person coalitions, the coincidence holds. Thus we can prove the main result after an adjustment of the appointment game's characteristic function.

3.1 The cost saving function

Let $N \in \mathcal{N}$, $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}^N$, and $S \subseteq N$. Since the characteristic function $c_{\mathbf{e}}(S) = c(r_S)$, simply $c(S)$, of appointment game is concave, we will transform the function into the *cost saving function* v, which is defined as, for all $i \in S \subseteq N$,

$$v(S) \equiv \sum_{i \in S} c(i) - c(S) = \sum_{i \in S^0} 2c_i - \sum_{\substack{\{i,j\} \subseteq N^0 \\ : i \succ_{r_S} j}} c_{i,j} = \sum_{i \in S \setminus \{i_{S_k}\}} \{c_i + c_{i+1} - c_{i,i+1}\}.$$

A game is *convex* if for each $S, T \in N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. The cost saving game (N, v) is convex. Here, i_{S_k} is the last sponsor in the connected set $S_k \subseteq S$. From the definition, we get $v(\{i\}) = \sum_{i \in \{i\}} c(i) - c(\{i\}) = 0$.

From now, we will work on the domain $\mathcal{V}_{\mathcal{E}_T}$ so that $v(S)$ means $v_{\mathbf{e}}(S)$.

Proposition 1. *On the domain $\mathcal{V}_{\mathcal{E}_T}$, the Shapley value and the Nucleolus coincide in the appointment game (N, c) if the Shapley value and the Nucleolus coincide in the cost saving game (N, v).*

Proof. It suffices to show that if the Nucleolus and the Shapley value corresponding to the cost saving function v are the same, the Nucleolus and the Shapley value corresponding to the cost function c in the appointment game will also be equal.

STEP 1. Let the cost saving game v.

$$v(S) \equiv \sum_{i \in S} c(i) - c(S)$$

(1) Given value function v of the cost saving game, with an allocation $x \in \mathbb{R}^N$, the excess for a coalition S is $e_S(v, x) \equiv \sum_{i \in S} x_i - v(S)$. Let the excess vector of v be $\theta(v, x) = (e_{S_1}(v, x), e_{S_2}(v, x), \cdots, e_{S_k}(v, x))$ where $k = 2^n - 2$. The imputation set of v is $X(v) = \{x \in \mathbb{R}^N | \sum_i x_i = v(N)\ and\ x_i \geq v(i)$ for all $i\}$. Given $x, y \in \mathbb{R}^N$, $x \leq_{(lex)} y$ means that x is lexicographically less than or equal to y. Then the nucleolus of (N, v) is defined to be

$$Nuc(N, v) = \{y | \theta(v, y) \leq_{(lex)} \theta(v, x), \text{for all } x \in X(v)\}.$$

(2) Given cost function c of the appointment game, with an allocation $x \in \mathbb{R}^N$, the excess for a coalition S is $e_S(c,x) \equiv c(S) - \sum_{i \in S} x_i$. The excess vector of c be $\theta(c,x) = (e_{S_1}(c,x), e_{S_2}(c,x), \cdots, e_{S_k}(c,x))$ where $k = 2^n - 2$. The imputation set of c is $X(c) = \{x \in \mathbb{R}^N | \sum_i x_i = c(N) \ and \ x_i \leq c(i) \text{ for all } i\}$. Then the nucleolus of (N,c) is defined to be

$$Nuc(N,c) = \{y | \theta(c,y) \leq_{(lex)} \theta(c,x), \text{ for all } x \in X(c)\}.$$

From the definition of $v(S)$, we can find the relation between the excess of the cost saving game $e_S(v,x)$ and the excess of the cost function of the appointment game $e_S(c,x)$. For each $i \in S \subseteq N$ with the allocation $x \in X(v)$, define the *cost* $z_i \equiv c(i) - x_i$. Then for each $S \subseteq N$,

$$\begin{aligned} e_S(v,x) &= \sum_{i \in S} x_i - v(S) = \sum_{i \in S} x_i - \{\sum_{i \in S} c(i) - c(S)\} \\ &= c(S) - \sum_{i \in S} \{c(i) - x_i\} = c(S) - \sum_{i \in S} z_i = e_S(c,z). \end{aligned}$$

An allocation rule is *efficient* if for an allocation $x \in X(v)$, $v(N) = \sum_{i \in N} x_i$ holds. The cost allocation z is efficient if the allocation $x \in X(v)$ is efficient.

$$\sum_{i \in N} z_i = \sum_{i \in N} c(i) - \sum_{i \in N} x_i = \sum_{i \in N} c(i) - v(N) = c(N).$$

Also, if an allocation x is in the imputation set $X(v)$, then cost z belongs to $X(c)$. By definition of the cost saving game v, we know that $v(\{i\}) = 0$ so that for $x \in X(v)$, each $x_i \geq v(i)$, that is, $x_i \geq 0$. This implies that inequality $z_i = c(i) - x_i \leq c(i)$ always holds. Therefore, if an allocation x belongs to $X(v)$, then the allocation z belongs to $X(c)$.

Now, we can say that if an allocation x is the Nucleolus of (N, v), then the cost allocation $z \equiv \{z_i = c(i) - x_i$ for all $i \in N\}$ is the Nucleolus of (N, c).

STEP 2. The shapley value ϕ for each N, v and i is given by,

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}[v(S \cup \{i\}) - v(S)].$$

Also, the shapley value ϕ for each N, c and i is given by,

$$\phi_i(c) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}[c(S \cup \{i\}) - c(S)].$$

By definition of $v(S) = \sum_{i \in S} c(i) - c(S)$,

$$\begin{aligned} v(S \cup \{i\}) - v(S) &= (\sum_{j \in S \cup \{i\}} c(j) - c(S \cup \{i\})) - (\sum_{j \in S} c(j) - c(S)) \\ &= c(i) - c(S \cup \{i\}) + c(S) = c(i) - \{c(S \cup \{i\}) - c(S)\}. \end{aligned}$$

Therefore, we get $\phi_i(v) = c(i) - \phi_i(c)$ and $\phi_i(c) = c(i) - \phi_i(v)$.

From the results of Step 1 and Step 2, we get following conclusion. In the appointment game, if we prove that $Nuc(N, v) = \phi(N, v)$, we also get $Nuc(N, c) = \phi(N, c)$. Note that the allocation x of the appointment game on the domain $\mathcal{V}_{\mathcal{E}_T}$ belongs to the imputation set $X(c)$. $\square$

Example 1. Let $N = \{1, 2, 3, 4, 5\}$ and an appointment route be given by $r = \{0, 1, 2, 3, 0, 4, 5, 0\}$, and the appointment with sponsor 5 be canceled and the coalition is $S = \{1, 2, 3, 4\}$. Then the route for S is $r_S = \{0, 1, 2, 3, 0, 4, 0\}$,

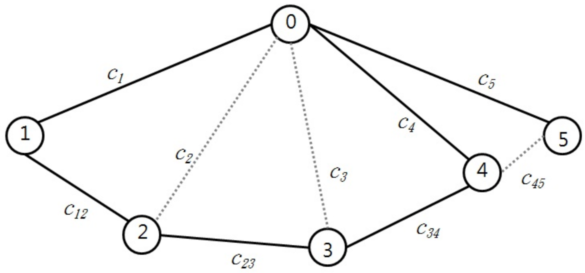

Figure 1: Example 1

and the cost and the cost saving for all connected sets(S_1 and S_2) in S is calculate as follow:

$$\begin{aligned} c(S) &= 2\sum_{n=1}^{4} c_i + \sum_{i \in S \setminus \{i_{S_k}\}} \{c_{i,i+1} - c_i - c_{i+1}\} \\ &= 2\sum_{n=1}^{4} c_i + (c_{1,2} - c_1 - c_2) + (c_{2,3} - c_2 - c_3) \\ &= c_1 + c_{1,2} + c_{2,3} + c_3 + 2c_4 \end{aligned}$$

$$\begin{aligned} v(S) &= \sum_{i \in S \setminus \{i_{S_k}\}} \{c_i + c_{i+1} - c_{i,i+1}\} \\ &= (c_1 + c_2 - c_{1,2}) + (c_2 + c_3 - c_{2,3}) \end{aligned}$$

Next, with **Proposition 1**, we will prove the coincidence of the Shapley value and the Nucleolus in the Appointment Game by using *the cost saving function* v.

3.2 Coincidence of the Shapley value and the Nucleolus in the Appointment Game

The coincidence of the Shapley value and the Nucleolus in *the cost saving game* can be easily proved by the method by *(Chun, Hokari(2007))*.

First, the basic setting of the queueing game and the two-person additive game will be introduced.

Let $N \equiv \{1, 2, \cdots, n\}$ be the set of agents. Each agent $i \in N$ costs his unit waiting cost, $\theta_i \geq 0$, and is assigned a position $\sigma_i \in N$ in a queue and a positive or negative transfer $t_i \in \mathbb{R}$. Each agent in σ_i^{th} position has own waiting cost of $(\sigma_i - 1)\theta_i$. A queueing problem is defined as a list $q = (N, \theta)$ where N is the set of agents and $\theta \in \mathbb{R}_+^N$ is the vector of unit waiting costs. Let Q^N be the class of all problems for N. An allocation for $q \in Q$ is a pair $x = (\sigma, t)$ where for each $i \in N$.

A set $S \subseteq N$ is a coalition. A game is a real-valued function v defined on all coalitions $S \subseteq N$ satisfying $v(\emptyset) = 0$. The number $v(S)$ is the worth of S. Let Γ^N be the class of games with players set N. For each $S \subseteq N$, its worth $v_q(S)$ is defined by setting: $v_q(S) = -\sum_{t \in S}(\sigma_i - 1)\theta_i$, where $\sigma^* \in Eff(S, \theta_S)$, which minimize the total waiting cost, and $\theta_S = (\theta_i)_{i \in S}$. Since the game v_q is concave, for each $S \subseteq N$, the queueing cost game, $v_c(S) \equiv -v_q(S)$ is defined. Obviously v_c is a game in Γ^N. Moreover, v_c is convex and its nucleolus is well-defined.

Lemma 1. *(Chun, Hokari(2007))*

For each $q \in Q^N$, its queueing cost game v_c satisfies

(i) for each $i \in N$, $v_c(i) = 0$;

(ii) for each $S \subseteq N$ with $|S| \geq 2$, $v_c(S) = \sum_{\substack{T \subseteq S, \\ |T|=2}} v_c(T)$ and $v_c(S) \geq 0$.

Chun and Hokari(2007) generalized that the Shapley value and the Nucleolus coincide in the particular games that satisfy the two conditions, with the following three lemmas.

Let $\tilde{\Gamma}^N$ be the class of games satisfying the two conditions of **Lemma 1**(Chun, Hokari(2007)).

Lemma 2. *(Chun, Hokari(2007))*

For each $v \in \tilde{\Gamma}^N$ and each $i \in N$, $\phi_i(v) = \frac{1}{2}\sum_{\substack{S \subseteq N, \\ i \in S, |S|=2}} v(S)$.

Lemma 3. *(Chun, Hokari(2007))*
For each $v \in \tilde{\Gamma}^N$ and each $i \in N$, if $x_i \equiv \frac{1}{2}\sum_{\substack{S\subseteq N,\\ i\in S, |S|=2}} v(S)$, then for each $S \subseteq N$, $v(S) - \sum_{i\in S} x_i = v(N\backslash S) - \sum_{i\in N\backslash S} x_i$.

For each $v : 2^N \to \mathbb{R}^N$ with $\sum_{i\in N} x_i = v(N)$, and each $\alpha \in \mathbb{R}$, let $S_\alpha(v, x) \equiv \{S \in 2^N | S \neq \emptyset \text{ and } v(S) - \sum_{i\in S} x_i \geq \alpha\}$
A collection $\mathfrak{B} \subseteq 2^N$ of coalitions is strictly balanced on N if there exists a list $(\delta_S)_{S\in\mathfrak{B}}$ of positive weights such that for each $i \in N$.

$$\sum_{\substack{S\in\mathfrak{B}\\ S\ni i}} \delta_S = 1.$$

Lemma 4. *(Kohlberg(1971))*
For each $v \in \Gamma^N$ and each $x \in I(v)$, $x = Nuc(v)$ if and only if for each $\alpha \in \mathbb{R}$ with $S_\alpha(v, x) \neq \emptyset$, there exists $S \subseteq \{\{i\}|i \in N\}$ and $v(\{i\}) - x_i = 0$ such that $S_\alpha(v, x) \cup S$ is strictly balanced on N.

These lemmas give the following theorem.

Theorem. *(Chun, Hokari(2007))* For each $v \in \tilde{\Gamma}^N$, $\phi(v) = Nuc(v)$.

Now, we are ready to prove the most important result in this paper.

Theorem 2. *Given economy $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}^N$, the Nucleolus and the Shapley value of appointment game $(N, c_{\boldsymbol{e}})$ coincide.*

Proof. The proof starts by implying the condition of (Chun, Hokari(2007)) to the cost saving game $v(S)$. Given an economy $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}^N$, we will show that the cost saving game $v(S)$ is in the class of $\tilde{\Gamma}^N$.

For each route of appointment r_s, its cost saving game $v(S)$ satisfies
(i) for each $i \in N$, $v(i) = 0$;
(ii) for each $S \subseteq N$ with $|S| \geq 2$, $v(S) = \sum_{\substack{T\subseteq S,\\ |T|=2}} v(T)$ and $v(S) \geq 0$.

The *condition (i)* is easily checked by the definition of $v(S)$, that is, $v(\{i\}) = \sum_{i\in\{i\}} c(i) - c(\{i\}) = 0$.

Condition (ii) holds, since the value of $v(T)$ for each pair $\{i,j\} \in N$ is defined as,

$$v(\{i,j\}) = \begin{cases} c_i + c_j - c_{i,j} & \text{, if } i \succ_{r_S} j, \\ 0 & \text{, otherwise.} \end{cases}$$

Precisely, we can derive the given equation in (ii) from the original definition of the cost saving game (N, v).

$$v(S) = \sum_{i\in S\setminus\{i_{S_k}\}} \{c_i + c_{i+1} - c_{i,i+1}\} \equiv \sum_{\substack{T\subseteq S,\\ |T|=2}} v(T) = \sum_{\substack{i,j\in T\subseteq S,\\ |T|=2,\\ i\succ_{r_S} j}} v(T) + \sum_{\substack{i,j\in T\subseteq S,\\ |T|=2,\\ i\not\succ_{r_S} j}} v(T)$$

$$= \sum_{\substack{i,j\in T\subseteq S,\\ |T|=2,\\ i\succ_{r_S} j}} v(T) = \sum_{\substack{i,j\in T\subseteq S,\\ |T|=2,\\ i\succ_{r_S} j}} \{c_i + c_j - c_{i,j}\}.$$

Now we know that the cost saving function $v(S)$ satisfies two conditions in **Lemma 1**. We already showed that $v(S)$ is convex(in Proposition 1). Thus the game (N, v) is in the class of game $\tilde{\Gamma}^N$. According to Lemma 2, Lemma 3, and Lemma 4, we accomplish the final result in *the cost saving game*. For each cost saving function $v \equiv v_{\mathbf{e}} \in \mathcal{V}_{\mathcal{E}_T}, \phi(v) = Nuc(v)$. Eventually, by Proposition 1, the coincidence of the Shapley value and the Nucleolus coincide in *the appointment game*. $\square$

Here, we have some examples showing **Lemma 1** and **Lemma 2** for an appointment game.

Example 2. Let $S = \{1,2,3,4\}$. For all $T \subseteq S, |T| = 2$, the cost saving function $v(T)$ is determined as $v(\{1,2\}) = \{c_1 + c_2 - c_{1,2}\}$, $v(\{2,3\}) = \{c_2 + c_3 - c_{2,3}\}$,

$v(\{3,4\}) = \{c_3 + c_4 - c_{3,4}\}$ and $v(\{1,3\}) = v(\{1,4\}) = v(\{2,4\}) = 0$ since $1 \not\succ_{r_S} 3,\ 1 \not\succ_{r_S} 4,\ 2 \not\succ_{r_S} 4$.

$$v(S) = v(\{1,2\}) + v(\{1,3\}) + v(\{1,4\}) + v(\{2,3\}) + v(\{2,4\}) + v(\{3,4\})$$
$$= v(\{1,2\}) + v(\{2,3\}) + v(\{3,4\})$$
$$= \{c_1 + c_2 - c_{1,2}\} + \{c_2 + c_3 - c_{2,3}\} + \{c_3 + c_4 - c_{3,4}\}$$

The next example shows that the Shapley value of *the appointment game* is derived from *the cost saving function* and **Lemma 2**.

Example 3. Let $S = \{1,2,3,4,5\}$, and the appointment route be given by $r_S = (0,1,2,3,4,0,5,0)$. Now we can indirectly get the Shapley value of the appointment game applying the cost saving function to **Lemma 2**.

i) $Sh_1(v) = \frac{1}{2}(c_1 - c_{1,2} + c_2)$, so that
$Sh_1(c) = 2c_1 - \frac{1}{2}(c_1 - c_{1,2} + c_2) = \frac{3c_1 + c_{1,2} - c_2}{2}$.

ii) $Sh_2(v) = \frac{1}{2}((c_1 - c_{1,2} + c_2) + (c_2 - c_{2,3} + c_3)) = \frac{1}{2}(c_1 - c_{1,2} + 2c_2 - c_{2,3} + c_3)$,
$Sh_2(c) = 2c_2 - \frac{1}{2}(c_1 - c_{1,2} + 2c_2 - c_{2,3} + c_3) = \frac{1}{2}(2c_2 + c_{1,2} + c_{2,3} - c_1 - c_3)$.

iii) $Sh_3(v) = \frac{1}{2}(c_2 - c_{2,3} + 2c_3 - c_{3,4} + c_4)$, so that
$Sh_3(c) = 2c_3 - \frac{1}{2}(c_2 - c_{2,3} + 2c_3 - c_{3,4} + c_4) = \frac{1}{2}(2c_3 + c_{2,3} + c_{3,4} - c_2 - c_4)$.

iv) $Sh_4(v) = \frac{1}{2}(c_3 - c_{3,4} + c_4)$, so that
$Sh_4(c) = 2c_4 - \frac{1}{2}(c_3 - c_{3,4} + c_4) = \frac{3c_4 + c_{3,4} - c_2}{2}$.

v) $Sh_5(v) = 0$, so that $Sh_5(c) = 2c_5$.

In fact, the same result of the Shapley value can be obtained in Proposition 1(Yengin(2012)). For each $N \in \mathcal{N}$, each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}^N$, and each $i \in N$,

- if $0 \succ_r i \succ_r 0$,then

$$SV_i(v_{\mathbf{e}}) = 2c_i.$$

- if there is $j \in N\backslash\{i\}$ such that either $0 \succ_r i \succ_r j$ or $j \succ_r i \succ_r 0$, then

$$SV_i(v_{\mathbf{e}}) = \frac{3c_i + c_{i,j} - c_j}{2}.$$

- if there is a pair $\{h, k\} \subseteq N\backslash\{i\}$ such that $h \succ_r i \succ_r k$, then

$$SV_i(v_{\mathbf{e}}) = \frac{1}{2}(2c_i + c_{h,i} + c_{i,k} - c_h - c_k).$$

Remark. Given cost saving game (N, v), the Nucleolus and the Shapley value also coincide with the τ-value(Chun, Hokari(2007)).

For each $v \in \Gamma^N$ and each $i \in N$, let $M_i(v) \equiv v(N) - v(N\backslash\{i\})$ and $m_i(v) \equiv v(\{i\})$. Then, the τ-value(Tijs 1987) selects the maximal feasible allocation on the line connecting $M(v) \equiv (M_i(v))_{i\in N}$ and $m(v) \equiv (m_i(v))_{i\in N}$.

τ-value, τ: For each convex game v,

$$\tau(v) \equiv \lambda M(v) + (1 - \lambda)m(v),$$

where $\lambda \in [0, 1]$ is chosen so as to satisfy

$$\sum_{j\in N}[\lambda(v(N) - v(N\backslash\{j\})) + (1 - \lambda)v(\{j\})] = v(N).$$

For a cost saving game v, $m(v) = 0$. Also, for each $j \in N$, $v(N) - v(N\backslash\{j\}) = \sum_{j\in S,|S|=2} v(S)$ and that $\lambda = \frac{1}{2}$. Therefore, the τ-value coincides with the Shapley value, and also the nucleolus in the cost saving games.

Also, the appointment game satisfies this coincidence with the τ-value. Since, the appointment game is concave, let $\tau(c)$ be

$$\tau(c) \equiv -\lambda M(c) - (1-\lambda)m(c),$$

where $\lambda \in [0,1]$ is chosen so as to satisfy

$$\sum_{j\in N}[-\lambda(c(N) - c(N\backslash\{j\})) - (1-\lambda)c(\{j\})] = -c(N).$$

For an appointment game c, and for each $j \in N$,

$$\sum_{j\in N}[\lambda(c(N) - c(N\backslash\{j\})) + (1-\lambda)c(\{j\})] =$$

$$\lambda(\sum_{\substack{\{i,j\}\in N\\ i\succ_T j}} 2c_{i,j} - \sum_{\substack{k\in N\\ i\succ_T k\succ_T j}} 2c_k) + (1-\lambda)(\sum_{i\in N} 2c_i) = c(N).$$

Therefore, $\lambda = \frac{1}{2}$ holds. From the definition of $v(S) = \sum_{i\in S} c(i) - c(S)$, we can derive $\tau_j(c) = -\frac{1}{2}(c(N) - c(N\backslash\{j\}) + c(j)) = \frac{1}{2}(v(N) - v(N\backslash\{j\})) = \tau_j(v)$. That is, the τ-value in the cost saving game and the appointment game coincide. This gives us an expanded result that the Shapley value, the nucleolus, and the τ-value coincide in the appointment game.

4 Concluding remarks

In this paper, we have shown that the Shapley value, the nucleolus, and the τ-value coincide in the fixed-route traveling game. The game has a good condition, that it has a non-empty core. Also, the Shapley value can be easily represented and the nucleolus is easily obtained. It can also be handled as a *2-person additive game*. However, the condition that the traveler goes back to her home or waits until the next appointment starts when an appointment is canceled seems to be restrictive in TSG. Therefore, more generalized model settings are required.

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국문초록

박 나 리
경제학부 경제학 전공
서울대학교 대학원

약속순서가 정해진 고정 경로를 여행하는 외판원문제(Fixed-route traveling salesman problem with appointments), 약칭(略稱)하여 약속 문제(Appointment problems)에서는 다음과 같은 상황을 상정한다. 여행자는 자신의 집에서 출발하여 다시 집으로 돌아오는데, 주어진 스폰서 집합 내에서 일정이 짜인 여행을 한다. 만약 약속된 경로 상의 한 스폰서가 약속을 취소하면, 여행자는 자신의 집으로 돌아온 후에 집에서부터 다음 약속 장소로 이동한다. 여행자의 여행비용은 방문을 받은 스폰서들이 부담하게 된다. 이 때, 협조게임이론에서 개발된 분배규칙을 적용하여 스폰서들에게 약속문제의 총비용을 배분하는 방식에 대해 논의한다. 이 논문에서는 위와 같은 클래스의 문제에 대해 협조게임이론에서 잘 알려진 배분규칙인 섀플리 밸류(Shapley value), 중핵(Nucleolus), 타우-밸류(τ-value)가 일치함을 보인다.

주요어 : 고정 경로 외판원문제, 약속 문제, 일치, 섀플리 밸류, 중핵, 타우-밸류.

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